Math 10B with Professor Stankova
Worksheet, Discussion #28; Tuesday, 5/7/2019
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1 Review Topics

1.1 Counting and Probability

1.
- Permutations and Combinations
  - Binomial coefficients
  - Poker-type problems
  - MISSISSIPPI problems
- Principle of Inclusion-Exclusion (PIE)
  - Complementary Counting
- Pigeonhole principle
- 12-Fold way
  - (In)distinguishable balls and (in)distinguishable boxes
  - Sterling Numbers of the Second kind
  - Partition Numbers
- Probability, Expected Value, Variance, Covariance
  - Random variable picture
  - Probability Mass Functions (PMF)
  - Formulas for expected values
  - Conditional probability
  - Independence
  - Chebyshev's Inequality
- Bayes' Theorem
- Distributions
  - Uniform Distribution
  - Bernoulli Trials
  - Binomial distribution
  - Hyper-geometric distribution
  - Geometric distribution
  - Poisson distribution
  - Normal distribution
  - t distribution

  - χ^2 distribution
  - Expected value and variance of each
- Maximum Likelihood
- Hypothesis Testing
  - Z Testing
  - t Testing
  - χ^2 Testing
  - Independence Testing
  - Null/Alternative Hypotheses
  - Type 1/type 2 errors, significance level, power
- Estimators and Confidence Intervals
  - Estimators for the mean and standard deviation
    * Biased/unbiased estimators
  - 95% confidence intervals

1.2 Miscellaneous

- Geometric sequences
- Induction
- Sorting Algorithms
  - Bubble Sort
  - Quick Sort
- Stable-matching algorithm
- Linear Regression
  - Least Squares Error
  - Finding line of best fit
- Correlation
- Gamma Function

2 Counting

2. True False There are as many bit strings of length n as there are subsets of a set of n elements.
3. True False Complementary counting is always easier than the method of exhaustion.
4. True False I have socks of 3 different colors. If I randomly pick 2 of them, then PHP tells me that at least 2 of them must be the same color.

5. True False We have $\binom{n}{r} = \binom{n}{n-r}$.

6. How many ways are there to rearrange the letters of $ZVEZDA$?

7. How many ways can I split up 20 distinct people into 4 identical groups of 5?

8. How many bit strings of length 10 begin with a 1 or end with a 1?

9. What is the coefficient of $x^2y^3z^5$ in $(x/2 + 3y - 2z)^{10}$? What about the coefficient of $x^3y^3z^3$?

10. How many ways are there fore 8 men and 5 women to stand in a line so that no two women stand next to each other?

11. How many ways can we select 5 elements from a set of 3 elements if order matters and repetition is allowed?

12. How many license plates of 3 letters followed by 3 numbers do not contain three of the same letters nor three of the same digit?

13. How many ways can I choose 8 donuts from a box containing 20 identical glazed donuts and 30 identical chocolate ones?

14. Prove that $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$ in two different ways.

15. How many ways can I create a license plate that has 3 letters followed by 3 numbers if I want exactly 1 I and at least 1 1.

16. How many ways can I pack my 10 groceries into 3 identical bags if the bags must be non-empty?

17. There are 2504 students at a school. Of them 1876 have taken Algebra, 999 have taken Biology, and 345 have taken chemistry. Moreover, 876 have taken both algebra and bio, 231 have taken bio and chem, and 290 have taken algebra and chem. Finally, 189 of them have taken all three. How many students have taken none of the three?

18. How many 5 card hands out of a standard 52 card deck have 4 different suits?

19. I am baking cookies for Alice Bob and Carol. Each want at least 1 cookie but Bob wants at least 3 cookies. Alice is on a diet and wants at most 3 cookies. How many ways can I divide the 10 cookies I made amongst them?

20. In the previous cookie problem, let X be the most number of cookies any one of Alice, Bob, or Carol gets (if they got $2, 3, 5$ cookies, then $X = 5$), what can we say about the minimum X can be?

3 Probability

21. True False If there are only n possible outcomes in Ω, then there are 2^n inputs to the probability function P.

22. True False For any two events we have $P(A|B) = P(B|A)$.

23. True False For any two events A, B, we have $P(A \cup B) = P(A) + P(B)$.

24. True False If A, B are events with $P(A), P(B) > 0$, then $P(A|B) > P(A)$.

25. True False If A, B are independent events, then $P(A|B) = P(A)$.

26. True False If A, B are independent events, then $\bar{A}$ and B are independent.

27. True False Two random variables X, Y are independent if for all a, b we have $P(XY = ab) = P(X = a) \cdot P(Y = b)$.

28. Let X be a random variable on a probability space Ω with a probability function P and let f be the PMF for X. Draw a picture of how all these variables interact and explain any special arrows that you have in your diagram. Do the same for if X is a continuous random variable.

29. In Berkeley, it is either sunny or cloudy. Suppose that it is cloudy with probability 30% and when it is cloudy, it rains with probability 80%. The probability that it rains with it is sunny is 1%. What is the probability that it rains in Berkeley?

30. Suppose that our outcome space is $\{1, 2, 3, 4\}$ with $P(\{1\}) = P(\{3\}) = 16\%, P(\{2\}) = 4\%, P(\{4\}) = 64\%$. Are $\{1, 2\}$ and $\{2, 3\}$ independent?

31. Suppose you play a lottery where 6 numbers are selected out of the numbers 1 to 40 inclusive. You pick 6 numbers, what is the probability that only one of them is correct?

32. Eve has 5 cards in her hand and I know that one of them is the ace of spades. What is the probability that she has a pair of aces (exactly 2 aces)?

33. You ask two friends for a favor. The first friend has a 10% chance of saying yes and the second has a 5% chance. The probability that they both say yes is 3%. What is the probability that at least one friend will help you?

34. A red-green colorblind person picks an apple out of a bag. There are 4 red apples and 1 green apple. With probability 3/4 he says the correct color of the apple he picked out. What is the probability that he says that the apple he picks out is red?

35. In the previous apple problem, what is the probability that the apple is actually red when he says it is red?

36. You have a bag containing 3 fair 6 sided die and one biased one so that the probability of rolling 1 is 0.5 and rolling $2-6$ each is 10%. You randomly select a die and roll the die to get a 1. What is the probability you selected a fair die?

4 Discrete Distributions

37. True False Doing n independent Bernoulli trials and counting the number of successes is a geometric distribution.

38. True False If X is a Poisson variable with parameter 1 and Y is Poisson with parameter 2, then the probability that $Y = 2$ is larger than the probability $X = 2$.

39. True False If $\mu = E[X]$, then $P(X = \mu)$ is higher than $P(X = k)$ for all other k.

40. True False If X, Y are random variables with $Cov(X, Y) = 0$, then X, Y are independent.

41. True False A constant random variable has variance 0.

42. True False If X has standard error σ and expected value μ, then $10X$ has expected value 10μ and standard error 10σ.

43. True False It is possible to have a Poisson random variable with expected value 1 and variance 2.

44. Suppose that I have a weighted die that lands on $1, 2, 3, 4, 5$ with equal probability and 6 5 times as likely as 1. Let X be the value of the die. What is the PMF for X?

45. A detective is gathering information about a bank robbery by interviewing citizens. Suppose that out of the 300 citizens in the town, 20 of them witnessed the crime. What is the probability that the detective interviews exactly 3 witnesses if she interviews 50 random distinct people?

46. For my weighted die in the previous problem, what is the probability in 10 rolls, I roll a 5 or 6 exactly 6 times? What kind of distribution is this?

47. For my weighted die in the previous problem, suppose that I keep rolling until I roll a 5 or 6. What is the expected number of times I need to roll the die? What kind of distribution is this?

48. Suppose that X is binomially distributed with $E[X] = 15$ and $Var(X) = 6$. How many trials n are there and what is the probability p of success?

49. Suppose the number of frogs seen at a pond each day is Poisson distributed with an average of 0.1 per day. What is the probability you see more than 1 frog?

50. Suppose that the number of students who fill out course evaluations per day is Poisson distributed and on average 2 students fill out evaluations per day. What is the probability that in a week, no students fill out evaluations? What is the probability that in a week, 7 people fill out evaluations?

51. When I roll a fair 6 sided die 10 times, what is the expected number of distinct numbers that appear? (For instance, if I roll $1, 1, 3, 3, 2$, there are 3 distinct numbers that appear)

52. Let $X_1, \ldots, X_4$ be i.i.d Bernoulli trials with $p = \frac{3}{4}$. Let $\bar{X}$ be the average of them. What is $Var[\bar{X}]$? Find $Cov(X_1, \bar{X})$ (Hint: Write $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$).

5 Miscellaneous

53. Use MMI to prove that $1 + \frac{1}{4} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n > 1$.

54. Let $a_{n+1} = a_n + 2a_{n-1}$ with $a_0 = 2$ and $a_1 = 1$. Use MMI to prove that $a_n = 2^n + (-1)^n$ for all $n \geq 0$.

55. Suppose that three people randomly pick a hat. What is the expected value of the number of people who choose their hat? (with proof). What is the variance?